$\int_0^L f(x)dx = \int_C^{C+L} f(x)dx$

PART I : ELEMENTARY CONCEPTS

Problem 1. (**5 points**)

Consider the differential equation

$(x^3 - x^2)y'' + (x^2 - 2x)y' + 4y = 0$

Find the ordinary points, regular singular points and irregular singular points(if there's any).

Problem 2. (**3 points**) Given a recurrence equation $a_{n+2} = \dfrac{-a_n}{2(n+2)}$, $n \geq 0$ and $a_0 = 1$, $a_1 = 0$.

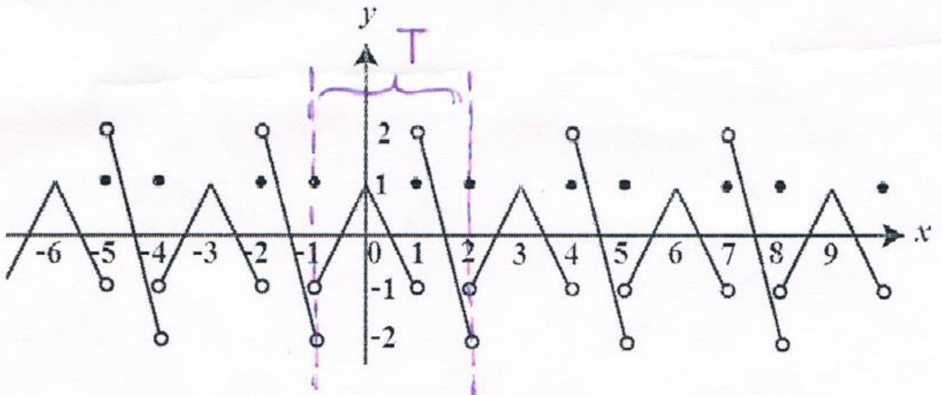
Then $a_{2558} - a_{2015} =$

Problem 3. (**3 points**)

3.1 (**2 points**) The elementary period of the function $f(x) = \cos\left(\frac{x}{4}\right) + \sin\left(\frac{x}{8}\right)$ is 16π.

$T_1 = \frac{2\pi}{\left(\frac{1}{4}\right)} = 8\pi \qquad T_2 = \frac{2\pi}{\left(\frac{1}{8}\right)} = 16\pi$

3.2 (**1 point**) Let the graph of a function $f(x)$ be given as follows.



The elementary period of the function $f(x)$ is 3.

Problem 4. (**12 points**) Let $f(x)$ be a periodic function with period 2. The Fourier series of f is given by

$\hookrightarrow T = 2,\ L = 1$

$$f(x) \sim \frac{3}{2} - \frac{6}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{\pi(2n-1)^2}\cos(2n-1)\pi x + \frac{(-1)^{n+1}}{n}\sin n\pi x\right]$$

$\frac{a_0}{2} = \frac{3}{2} \to a_0 = 3$

$a_n = -\frac{6}{\pi}\frac{2}{\pi(2n-1)^2}$

$b_n = -\frac{6}{\pi}\frac{(-1)^{n+1}}{n}$

4.1 $b_{2014} = \dfrac{6}{2015\pi}$

4.1) $b_n = -\frac{6}{\pi}\frac{(-1)^{n+1}}{n} \to b_{2014} = -\frac{6}{\pi}\frac{(-1)^{2015}}{2015}$

4.2 $\int_{-1}^{1} f(x)\cos 3\pi x\,dx = La_n = La_2 = -\frac{4}{3\pi^2}$

$3\pi = (2n-1)\pi$
$n = 2$

4.2) $2n-1 = 3 \to n = 2$

$a_n = -\frac{6}{\pi}\left[\frac{2}{\pi(2n-1)^2}\right]$

$a_2 = -\frac{4}{3\pi^2}$

4.3 $\int_{-4}^{4} f(x)\sin 4\pi x\,dx = 4\int_{-1}^{1} f(x)\sin 4\pi x\,dx$

$\hookrightarrow 4\pi = n\pi$
$n = 4$

$= 4Lb_n = 4b_4$

$= \frac{6}{\pi}$ #

$\int_{-4}^{4} = \int_{-4}^{0} + \int_{0}^{4} = \int_{-2}^{2} \int_{-2}^{2}$

4.4 $\int_{-4}^{4} f(x+4)dx =$ $\int_{-4}^{4} f(x)\,dx = 4\int_{-1}^{1} f(x)\,dx = 4 L a_0 = 4(3) = 12$ #

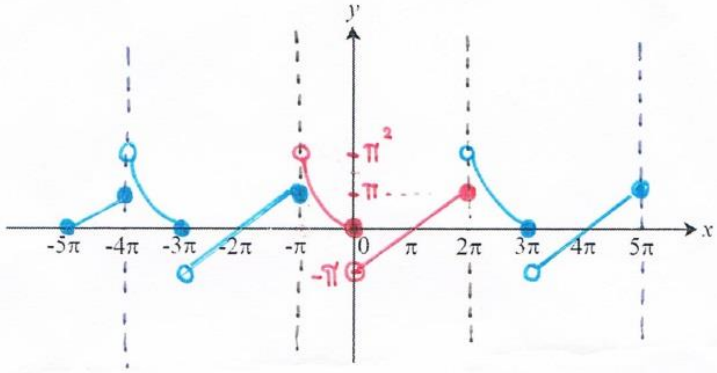
$x=4$
$x=-4$

4.5 $\int_{-4}^{4} f(2x)dx =$ $\int_{x=-4}^{x=4} f(2x)\,dx = \int_{-8}^{8} f(x)\frac{dx}{2} = \frac{8}{2}\int_{-1}^{1} f(x)\,dx = 12$ #

4.6 $\int_{-4}^{4} \sin^{2014} 3\pi x \sin 5\pi x dx =$ 0

Problem 5. (2 points) Let f be a function given by $f(x) = \begin{cases} x^2, & -\pi < x \le 0; \\ x - \pi, & 0 < x \le 2\pi \end{cases}$

and $f(x + 3\pi) = f(x)$. Sketch the graph of $f(x)$ for $x \in [-5\pi, 5\pi]$.



Problem 6. (6 points) Let

$$f(x) = \begin{cases} 1, & -1 \le x < 0; \\ 2x, & 0 \le x < 1. \end{cases} \qquad T=2,\ L=1$$

and $f(x + 2) = f(x)$ for all $x \in \mathbb{R}$. Suppose that the Fourier series of f is

$$\frac{3}{4} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x, \text{ for all } x \in \mathbb{R}.$$

$\frac{a_0}{2} = \frac{3}{4} \to a_0 = \frac{3}{2}$

6.1 $a_0 =$ $\frac{3}{2}$

6.2 $\sum_{n=1}^{\infty} a_n =$ $\frac{-1}{4}$

6.3 $\sum_{n=1}^{\infty} (-1)^n a_n =$ $\frac{5}{4}$

(6.2) $\boxed{x=0}$; $\frac{f(0^+)+f(0^-)}{2} = \frac{3}{4} + \sum_{n=1}^{\infty} a_n$

$\frac{0+1}{2} = \frac{3}{4} + \sum_{n=1}^{\infty} a_n$

$\sum_{n=1}^{\infty} a_n = \frac{1}{2} - \frac{3}{4} = \frac{-1}{4}$ #

6.1) $\frac{a_0}{2} = \frac{3}{4} \to a_0 = \frac{3}{2}$

6.2) $\boxed{x=0}$: $\frac{f(0^+)+f(0^-)}{2} = \frac{3}{4} + \sum_{n=1}^{\infty} a_n \cos 0 + \cancel{b_n \sin 0}$ 0

$\frac{0+1}{2} = \frac{3}{4} + \sum_{n=1}^{\infty} a_n$

$\therefore \sum_{n=1}^{\infty} a_n = \frac{1}{2} - \frac{3}{4} = \frac{-1}{4}$

6.3) $\boxed{x=1}$: $f(1) = \frac{3}{4} + \sum_{n=1}^{\infty} a_n \cos n\pi$

$2 = \frac{3}{4} + \sum_{n=1}^{\infty} (-1)^n a_n$

$\sum_{n=1}^{\infty} (-1)^n a_n = 2 - \frac{3}{4} = \frac{5}{4}$ #

Problem 7. **(9 points)** Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π. Suppose that the Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

7.1 Find $\int_{-\pi}^{\pi} f(x+\pi)\,dx$ in terms of a_n and b_n.

7.2 Find $\int_{-\pi}^{\pi} f(x)\cos^2 x\,dx$ in terms of a_n and b_n.

7.3 Find the Fourier series of $f(x+\frac{\pi}{2})$. Write the Fourier coefficients in terms of a_n and b_n.

Problem 8. **(10 point)** **True of False**.
(correct 2 points, no answer -0.5 point, wrong -1 point)

$L = 1$ $\quad a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx$

8.1 F If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x$, then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$.

8.2 T If f is an odd function and g is an even function, then $\int_{-\pi}^{\pi} f(x)g(x)dx = 0$.

8.3 T If f is a periodic function with period 4 then $g(x) = f(2x+1)$ is a periodic function with period 2.

8.4 F If f and g are odd functions, then the Fourier series of $f+g$ contains cosine functions only. $f+g \to$ odd $f^n \to$ Fourier sine.

8.5 T Let $p(x)y'' + q(x)y' + s(x)y = 0$ be a differential equation of order 2 where $p(x)$, $q(x)$ and $s(x)$ are polynomials. Then this equation has an ordinary point.

$P(x) \neq 0$

$x = \frac{Ls}{\pi}$

$g(s) = f(x)$

PART II : PROBLEM SOLVING

$\rightarrow x^3-1=0 \rightarrow x=1$ singular

Problem 9. **(8 points)** Find the series solution of $(x^3-1)y'' + x^2y' + xy = 0$ about $x = 0$. ordinary ($\Leftarrow$ Please continue on the back of the previous page.)

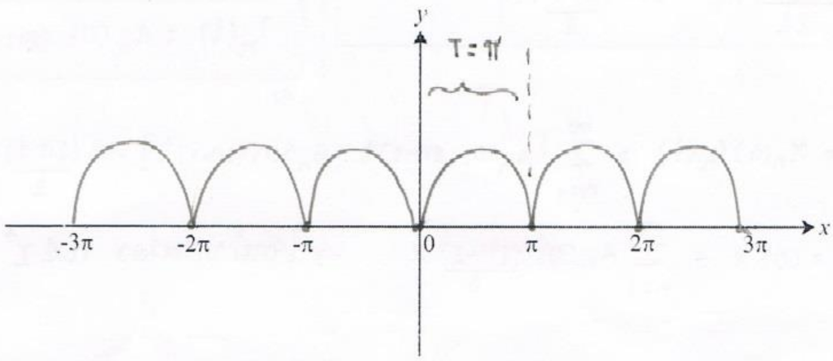
Problem 10. **(8 points)** Consider the differential equation $8x^2y'' + 10xy' + (x-1)y = 0$.
Suppose that the indicial equation is $(4r-1)(2r+1) = 0$
and the recurrence relation is $a_n = \dfrac{-a_{n-1}}{(4n+4r-1)(2n+2r+1)}$, $n \geq 1$.
Find the series solution of this equation about $x = 0$.

Problem 11. **(11 points)** Let f be a π-periodic function, where $f(x) = \sin x$ on $[0, \pi)$.

11.1 **(2 points)** Sketch the graph of f on $[-3\pi, 3\pi]$.

$T = \pi$

$L = \dfrac{\pi}{2}$



11.2 **(5 points)** It is known that the Fourier series of f is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Find a_0, a_n and b_n, $n = 1, 2, \ldots$.

$a_0 = \dfrac{1}{\pi/2}\displaystyle\int_0^{\pi} \sin x \, dx$

$= \dfrac{2}{\pi}\left[-\cos x\right]_0^{\pi} = -\dfrac{2}{\pi}\left[\overset{-1}{\cancel{\cos}\pi} - \overset{1}{\cancel{\cos}0}\right]$

$= \dfrac{4}{\pi}$ #

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x \cos nx \, dx$

$= -\dfrac{2}{\pi}\left[\dfrac{(-1)^{n+1}}{n+1} + \dfrac{(-1)^{n+1}}{1-n} - \dfrac{1}{1+n} + \dfrac{1}{1-n}\right]$

$= -\dfrac{2}{\pi}\left[\dfrac{(-1)^{n+1}-1}{1+n} + \dfrac{(-1)^{n+1}+1}{1-n}\right]$

$b_n = 0$ # even $f^{\underline{n}}$

$\sin(1+n)x + \sin(1-n)x$

$\downarrow \int$

$\left.-\dfrac{\cos(1+n)x}{1+n} - \dfrac{\cos(1-n)x}{1-n}\right|_0^{\pi}$

11.3 **(2 points)** At which points in $[-3\pi, 3\pi]$, $\rightarrow T = 6\pi$

$$f(x) = \text{ Fourier series of } f \text{ at } x.$$

$f(x+6\pi) = f(x) \rightarrow$ เหมือน 11.2

11.4 **(3 points)** Let $f(x) = \dfrac{2}{\pi} + \displaystyle\sum_{n=2}^{\infty} \frac{2(1+(-1)^n)}{\pi(1-n^2)} \cos nx$ and $f(n\pi) = 0$. Find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1-4n^2}$. $= \dfrac{1}{-3} + \dfrac{1}{-15} + \dfrac{1}{-35} + \dfrac{1}{-63} + \ldots$

$x = n\pi$: $\quad 0 = \dfrac{2}{\pi} + \displaystyle\sum_{n=2}^{\infty} \dfrac{2(1+(-1)^n)}{\pi(1-n^2)} \cos n^2\pi$

$0 = \dfrac{2}{\pi} + \Big[\underset{\uparrow\, n=2}{\dfrac{4}{\pi(-3)} \cos 4\pi} + \underset{\uparrow\, n=3}{0} + \underset{\uparrow\, n=4}{\dfrac{4\cos 16\pi}{\pi(-15)}} + \ldots\Big]$

$-\dfrac{2}{\cancel{\pi}} = \dfrac{4}{\cancel{\pi}}\left[\dfrac{1}{-3} + \dfrac{1}{-15} + \ldots\right]$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{1-4n^2} = -\frac{1}{2}$ #

Math III '57 (Final)

(14) Wave equation

Step 1 $\quad u(x,t) = X(x)T(t) = XT \quad , \quad 0<x<\overset{\to L}{\pi}$

หา $\quad u_{tt} = XT'' \quad , \quad u_{xx} = X''T$

แทนค่า $\quad XT'' = 4X''T$

$$\frac{X''}{X} = \frac{T''}{4T} = -\lambda$$

ODE $\quad X'' + \lambda X = 0$

BCs $\quad X'(0)=0 \quad , \quad X(\pi)=0$

Sturm-Liouville case 3

$$\boxed{\left.\begin{aligned} \lambda_n &= \left[\frac{(2n-1)\pi}{2L}\right]^2 = \frac{(2n-1)^2}{4} \\ X_n(x) &= \cos\frac{(2n-1)\pi x}{2L} = \cos\frac{(2n-1)}{2}x \end{aligned}\right\} n=1,2,3,\dots}$$

ODE $\quad T'' + 4\lambda T = 0$

$$m^2 + 4\lambda = 0$$

$$m^2 = -4\lambda \quad , \quad \lambda>0$$

$$m = \pm\sqrt{4\lambda}\, i$$

$$T = A\cos\sqrt{4\lambda}\,t + B\sin\sqrt{4\lambda}\,t$$

$$\boxed{T_n(t) = A_n\cos(2n-1)t + B_n\sin(2n-1)t}$$

step 2 $\quad u(x,t) = X_n(x)T_n(t) = \displaystyle\sum_{n=1}^{\infty}\left[\overset{?}{A_n}\cos(2n-1)t + \overset{?}{B_n}\sin(2n-1)t\right]\cos\frac{(2n-1)}{2}x$

ICs 1 $\quad u(x,0) = \cos x = \displaystyle\sum_{n=1}^{\infty} A_n\cos\frac{(2n-1)}{2}x \quad \to$ โจทย์บกพร่อง $n \notin I^+$

ICs 2 $\quad u_t(x,t) = \displaystyle\sum_{n=1}^{\infty}\left[-A_n(2n-1)\sin(2n-1)t + B_n(2n-1)\cos(2n-1)t\right]\cos\frac{(2n-1)}{2}x$

$$u_t(x,0) = 1 = \sum_{n=1}^{\infty} B_n(2n-1)\cos\frac{(2n-1)}{2}x \qquad \text{even } f^n$$

จะได้ $\quad B_n(2n-1) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} 1\cos\frac{(2n-1)}{2}x = \frac{2}{\pi}\left[\frac{\sin\frac{(2n-1)}{2}x}{\frac{(2n-1)}{2}}\right]_0^{\pi} = \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2} = (-1)^{n+1}\frac{4}{(2n-1)\pi}$

$$B_n = (-1)^{n+1}\frac{4}{(2n-1)^2\pi}$$

FS[3x+3] = FS[3x] + FS[3]
= 3 FS[x] + 3

Problem 12. **(11 points)** Let f be a continuous 2π-periodic function with Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where a_0, a_n and b_n, $n = 1, 2, \ldots$ are Fourier coefficients. Let

$$g(x) = \frac{f(x) + f(-x)}{2} \text{ and } h(x) = \frac{f(x) - f(-x)}{2}.$$

12.1 **(2 points)** Show that $f(x) = g(x) + h(x)$

12.2 **(3 points)** Show that g is an even function and h is an odd function.

12.3 **(3 points)** Find the Fourier cosine of g.

12.4 **(3 points)** Find the Fourier sine of h.

Problem 13. **(8 points)** Consider the Sturm-Liouville problem

$$X''(x) = 2\lambda X(x)$$

where $0 < x < 3$ and $X?(0) = 0; X'?(3) = 0$.
Find the eigenvalue λ_n and its corresponding eigenfunction $X_n(x)$.

($\Leftarrow$ Please continue on the back of the previous page.)

Problem 14. **(8 points)** Solve the partial differential equation

$$u_{tt}(x,t) = 4u_{xx}(x,t) \quad ; \quad 0 < x < \pi, \ t > 0$$
$$u_x(0,t) = 0 \text{ and } u(\pi,t) = 0 \quad ; \quad t > 0$$
$$u(x,0) = \cos x \text{ and } u_t(x,0) = 1 \quad ; \quad 0 < x < \pi$$

Remark : To solve the Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0, \ 0 \le x \le L,$$

students can use results from the following table.

Boundary Conditions	Eigenvalue λ_n	Eigenfunction $X_n(x)$	
$X(0) = 0 \quad X(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\sin \frac{n\pi x}{L}$	$n = 1, 2, \ldots$
$X'(0) = 0 \quad X'(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\cos \frac{n\pi x}{L}$	$n = 0, 1, 2, \ldots$
$X'(0) = 0 \quad X(L) = 0$	$\left(\frac{(2n-1)\pi}{2L}\right)^2$	$\cos \frac{(2n-1)\pi x}{2L}$	$n = 1, 2, 3, \ldots$
$X(0) = 0 \quad X'(L) = 0$	$\left(\frac{(2n-1)\pi}{2L}\right)^2$	$\sin \frac{(2n-1)\pi x}{2L}$	$n = 1, 2, 3, \ldots$

12.1) $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{2f(x)+0}{2} = f(x)$ #

12.2) Find $g(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(-x)+f(x)}{2} = g(x)$ $\therefore$ g is even f^{n}

Find $h(-x) = \frac{f(-x)-f(-(-x))}{2} = \frac{f(-x)-f(x)}{2} = -\left[\frac{f(x)-f(-x)}{2}\right] = -h(x)$

$\therefore$ h is odd f^{n}

12.3) $g(x) = \frac{1}{2}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \cancel{b_n \sin nx} + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \underbrace{\cos(-nx)}_{\cos(nx)} + b_n \underbrace{\sin(-nx)}_{\cancel{-\sin(nx)}}\right]$

$= \frac{1}{2}\left[a_0 + 2\sum_{n=1}^{\infty} a_n \cos nx\right]$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ #

12.4) $h(x) = \frac{1}{2}\left[\cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx - \left(\cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n \underbrace{\cos(-nx)}_{\cos nx} + b_n \underbrace{\sin(-nx)}_{-\sin nx}\right)\right]$

$= \frac{1}{2}\left[2\sum_{n=1}^{\infty} b_n \sin nx\right] = \sum_{n=1}^{\infty} b_n \sin nx$ #

Prob 7 $f(x+2\pi) = f(x)$ $\quad T = 2\pi \rightarrow L = \pi$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

7.1) $$\int_{-\pi}^{\pi} f(x+\pi)\,dx = \int_{x=-\pi}^{x=\pi} f(x+\pi)\,dx = \int_{x+\pi=-\pi}^{x+\pi=\pi} f(x+\pi+\pi)\,d(x+\pi)$$

$$= \int_{-2\pi}^{0} f(x+2\pi)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \pi a_0 = \frac{\pi}{2}\left[\frac{a_n}{\cos nx} + \frac{b_n}{\sin nx}\right]$$ #

7.2) $$\int_{-\pi}^{\pi} f(x)\cos^2 x\,dx = \int_{-\pi}^{\pi} f(x)\left[\frac{1+\cos 2x}{2}\right]dx = \int_{-\pi}^{\pi} \frac{1}{2} f(x)\,dx + \int_{-\pi}^{\pi} \frac{f(x)}{2}\underbrace{\cos 2x}_{n=2}\,dx$$

$$= \frac{1}{2}(\pi a_0) + \frac{1}{2}(\pi a_2) = \frac{\pi}{2}(a_0 + a_2) = \frac{\pi}{2}(a_0 + a_0\cos 2x)$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2}\left(\frac{a_n}{\cos nx} + \frac{b_n}{\sin nx}\right)(1+\cos 2x)\right]$$

$$= \frac{\pi^2}{4}\left(\frac{a_n}{\cos nx} + \frac{b_n}{\sin nx}\right)(1+\cos 2x)$$ #

7.3) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad x \in [-\pi, \pi]$$

$$f\left(x+\frac{\pi}{2}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\left(x+\frac{\pi}{2}\right) + b_n \sin n\left(x+\frac{\pi}{2}\right) \qquad x+\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \rightarrow L = \pi$$

$$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}+\frac{\pi}{2}}^{\frac{3\pi}{2}-\frac{\pi}{2}} f\left(x+\frac{\pi}{2}\right)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$\int x \sin ax\,dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

$$\int x \cos ax\,dx = \frac{x\sin ax}{a} + \frac{\cos ax}{a^2}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_0 \cos nx \rightarrow a_0 = \frac{a_n}{\cos nx}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = a_0 \sin nx \rightarrow a_0 = \frac{b_n}{\sin nx}$$

$$2a_0 = \frac{a_n}{\cos nx} + \frac{b_n}{\sin nx}$$

$$a_0 = \frac{a_n \sin nx + b_n \cos nx}{2\sin nx \cos nx}$$

$$a_0 = \frac{a_n \sin nx + b_n \cos nx}{\sin(2nx)}$$