Tutorial : Week XV Ordinary and Singular Points

1. (**Final** 56)

1.1 Give an example of a linear differential equation that $x = 1$ is a **singular point**.

$(x-1)y'' + xy' + 2y = 0$

1.2 According to the example in 1.1, whether $x = 1$ is a regular or an irregular singular point ? Verify your answer?

$\boxed{x=1}$

▸ $\lim_{x\to1} \cancel{(x-1)}\dfrac{x}{\cancel{(x-1)}} = 1$ exist

▸ $\lim_{x\to1} (x-1)^2\dfrac{2}{(x-1)} = \lim_{x\to1}(x-1)2 = 0$ exist

$\therefore$ $x=1$ is a regular singular point.

2. (**Final** 55) Consider the differential equation

$$(x^2 - 2x)y'' + (x-1)y' - 4y = 0.$$

Find (if there's any) the ordinary points, regular singular points, and irregular points.

(1) $x^2-2x \neq 0$

$x(x-2) \neq 0$

$x \neq 0, 2$

$\therefore$ $x \in \mathbb{R} - \{0,2\}$ เป็น ordinary points. #

(2) $\therefore$ $x = 0, 2$ เป็น singular points.

$\boxed{x=0}$

▸ $\lim_{x\to0} (x-0)\dfrac{(x-1)}{x^2-2x} = \lim_{x\to0}\dfrac{\cancel{x}(x-1)}{\cancel{x}(x-2)} = \dfrac{1}{2}$ exist

▸ $\lim_{x\to0} (x-0)^2\dfrac{-4}{x^2-2x} = \lim_{x\to0}\dfrac{-4x^{\cancel{2}\,x}}{\cancel{x}(x-2)} = \dfrac{0}{-2} = 0$ exist

$\therefore$ $x=0$ เป็น regular singular point. #

$\boxed{x=2}$

▸ $\lim_{x\to2} (x-2)\dfrac{(x-1)}{x^2-2x} = \lim_{x\to2}\dfrac{\cancel{(x-2)}(x-1)}{x\cancel{(x-2)}} = \dfrac{1}{2}$ exist

▸ $\lim_{x\to2} (x-2)^2\dfrac{-4}{x^2-2x} = \lim_{x\to2}\dfrac{-4\cancel{(x-2)}^{2\,(x-2)}}{x\cancel{(x-2)}} = \dfrac{0}{2}$ exist

$\therefore$ $x=2$ เป็น regular singular point. #

3. (**Final** 54) Consider the differential equation

$$(x+2)y'' + xy' - y = 0.$$

Find (if there's any) the ordinary points, regular singular points, and irregular points.

$p(x) = x+2 \qquad q(x) = x \qquad r(x) = -1$

(1) $x+2 \neq 0$

$x \neq -2$

$\therefore\ x \in \mathbb{R} - \{-2\}$ เป็น ordinary points.

(2) $x = -2$ เป็น singular point

check ▸ $\lim_{x \to -2} (x+2)\frac{x}{(x+2)} = \lim_{x \to -2} x = -2$ exist

▸ $\lim_{x \to -2} (x+2)^2 \frac{(-1)}{(x+2)} = \lim_{x \to -2} -(x+2) = 0$ exist

$\therefore\ x = -2$ เป็น regular singular points.

(1) $a_0 = 0$

$a_1 = 1$

$a_2 = 2$

$a_3 = 3$

$k=1$ $\quad a_4 = \dfrac{a_0}{2\cdot 6} = 0$ เริ่มเวียนเกิด

$a_5 = \dfrac{a_1}{2\cdot 7} = \dfrac{1}{2\cdot 7}$

$a_6 = \dfrac{a_2}{2\cdot 8} = \dfrac{2}{2\cdot 8}$

$a_7 = \dfrac{a_3}{2\cdot 9} = \dfrac{3}{2\cdot 9}$

$k=2$ $\quad a_8 = \dfrac{a_4}{2\cdot 10} = 0$

$a_9 = \dfrac{a_5}{2\cdot 11} = \dfrac{1}{2\cdot 11}\left(\dfrac{1}{2\cdot 7}\right) = \dfrac{1}{2^2}\left(\dfrac{1}{7\cdot 11}\right)$

$a_{10} = \dfrac{a_6}{2\cdot 12} = \dfrac{1}{2\cdot 12}\left(\dfrac{2}{2\cdot 8}\right) = \dfrac{1}{2^2}\left(\dfrac{2}{8\cdot 12}\right)$

$a_{11} = \dfrac{a_7}{2\cdot 13} = \dfrac{1}{2\cdot 13}\left(\dfrac{3}{2\cdot 9}\right) = \dfrac{1}{2^3}\left(\dfrac{3}{9\cdot 13}\right)$

$k=3$ $\quad a_{12} = \dfrac{a_8}{2\cdot 14} = 0$

$a_{13} = \dfrac{a_9}{2\cdot 15} = \dfrac{1}{2^3}\left(\dfrac{1}{7\cdot 11\cdot 15}\right)$

$a_{14} = \dfrac{a_{10}}{2\cdot 16} = \dfrac{1}{2^3}\left(\dfrac{2}{8\cdot 12\cdot 16}\right)$

$a_{15} = \dfrac{a_{11}}{2\cdot 17} = \dfrac{1}{2^3}\left(\dfrac{3}{9\cdot 13\cdot 17}\right)$

Run อนุกรมใหม่ ใช้ a_k จะได้ไม่มั่ว

$\underbrace{\hspace{6em}}_{k=1,2,\ldots}$

จะได้ $a_{4k} = 0$

$a_{4k+1} = \dfrac{1}{2^k}\cdot\left[\dfrac{1}{7\cdot 11\cdot 15\cdot\ldots\cdot(4k+3)}\right]$

$a_{4k+2} = \dfrac{1}{2^k}\cdot\left[\dfrac{2}{8\cdot 12\cdot 16\cdot\ldots\cdot(4k+4)}\right]$

$a_{4k+3} = \dfrac{1}{2^k}\left[\dfrac{3}{9\cdot 13\cdot 17\cdot\ldots\cdot(4k+5)}\right]$

$a_n = a_1 + (n-1)d$
$= 7 + (n-1)4$
$= 4n+3$

$8 + (n-1)4 = 4n+4$

$9 + (n-1)4 = 4n+5$

$\because$ 322271 หารด้วย 4 เหลือเศษ 3 $\leftarrow k$

$322271 = 4(80567) + 3$

Tutorial : Week XV Series Solutions about an Ordinary Point

1. **(Final** 56) Suppose $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_n = \dfrac{a_{n-4}}{2(n+2)}$, $n \geq 4$.

   Compute $a_{322271} = \dfrac{1}{2^{80567}}\left[\dfrac{3}{9\cdot 13\cdot 17\cdot \ldots \cdot 322273}\right]$.

2. **(Final** 56) Find the series solution of the differential equation

$$(1-x^2)y'' - 2xy' + 2y = 0 \rightsquigarrow y'' - x^2y'' - 2xy' + 2y = 0$$

about $x = 0$. ↗ is ordinary point.

step1 Set up $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} a_n x^n$

step 2 Diff $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

step3 Shift

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$+2$, $\to -2$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$n=0,1$ $\quad n=1 \quad n=0,1$

step 4 แยก

$$2a_2 + 6a_3x - \cancel{2a_1}x + 2a_0 + \cancel{2a_1}x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n\right]x^n = 0$$

Step5 $2a_2 + 2a_0 = 0 \to a_2 = -a_0 = -\frac{1}{1}a_0$ $\quad (n+2)(n+1)a_{n+2} + [2 - 2n - n(n-1)]a_n = 0$

$a_3 = 0$

$$a_{n+2} = \frac{(n^2+n-2)}{(n+2)(n+1)}a_n = \frac{\cancel{(n+2)}(n-1)\,a_n}{\cancel{(n+2)}(n+1)}$$

$$a_{n+2} = \frac{(n-1)}{(n+1)}a_n \quad , n \geq 2$$

$n=2$; $a_4 = \frac{1}{3}a_2 = \frac{1}{3}(-a_0) = -\frac{1}{3}a_0$

$n=3$; $a_5 = \frac{2}{4}a_3 = 0$

$n=4$; $a_6 = \frac{3}{5}a_4 = \frac{3}{5}\left(-\frac{1}{3}a_0\right) = -\frac{1}{5}a_0$

$n=5$; $a_7 = \frac{4}{6}a_5 = 0$

$n=6$; $a_8 = \frac{5}{7}a_6 = +\frac{5}{7}\left(-\frac{1}{5}a_0\right) = -\frac{1}{7}a_0$

$\vdots$

$a_{2k} = -\frac{1}{(2k-1)}a_0 \quad , k \geq 1$

step6 $y = \underbrace{a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots}_{\text{พจน์คี่ เป็น 0 เหลือ } a_0, a_1}$

$y = \sum_{n=1}^{\infty} a_{2n} x^{2n}$

$y = a_0 + a_1x - \sum_{n=1}^{\infty} \frac{1}{(2n-1)} a_0 x^{2n}$.

3.) $a_0 = 1$

$a_1 = 0$

$a_n = \dfrac{a_{n-2}}{n(n+3)}, \quad n \geq 2$

$n=2 \; : \; a_2 = \dfrac{a_0}{2\cdot5} = \dfrac{1}{2\cdot5}$

$n=3 \; : \; a_3 = \dfrac{a_1}{3\cdot6} = 0$

$n=4 \; : \; a_4 = \dfrac{a_2}{4.7} = \dfrac{1}{4.7}\left(\dfrac{1}{2\cdot5}\right)$

$n=5 \; : \; a_5 = \dfrac{a_3}{5.8} = 0$

$n=6 \; : \; a_6 = \dfrac{a_4}{6\cdot9} = \dfrac{1}{6.9}\cdot\dfrac{1}{4.7}\left(\dfrac{1}{2\cdot5}\right)$

$n=7 \; : \; a_7 = \dfrac{a_5}{7.10} = 0$

$5, 7, 9, \ldots \rightarrow a_k = 5 + (k-1)2 = 2k+3$

$2, 4, 6, \ldots \rightarrow a_k = 2k$

$$a_{2k} = \frac{1}{(2\cdot4\cdot6\cdot\ldots\cdot2k)(5\cdot7\cdot9\cdot\ldots\cdot(2k+3))} = \frac{1}{2^k\,k!\,(5\cdot7\cdot9\cdot\ldots\cdot(2k+3))}$$

$a_{2k-1} = 0$

$$\therefore \; a_{100} - a_{101} = a_{2(50)} - \cancel{a_{2(51)-1}}^{\,0}$$

$$= \frac{1}{2^{50}\cdot 50!\,[5\cdot7\cdot9\cdot\ldots\cdot103]}$$

3. (**Final** 55) Suppose $a_0 = 1$, $a_1 = 0$, $a_n = \dfrac{a_{n-2}}{n(n+3)}$, $n \geq 2$.

   Compute $a_{100} - a_{101} =$ $\dfrac{1}{2^{50} \cdot 50! \, [5 \cdot 7 \cdot 9 \cdot \ldots \cdot 103]}$.

4. (**Final** 55) Find the series solution of the differential equation

$$y'' + xy' + (x^2 + 2)y = 0 \qquad y'' + xy' + x^2y + 2y = 0$$

   about $x = 0$. $\rightsquigarrow$ ordinary

Step 1 $\quad y = \sum_{n=0}^{\infty} a_n x^n$

Step 2 $\quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Step 3 $\quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} + \sum_{n=0}^{\infty} 2a_n x^n = 0$

$\quad \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$

$\quad n=0,1 \qquad n=1 \qquad n=0,1$

Step 4 $\quad \underbrace{2a_2 + 6a_3 x + a_1 x + 2a_0 + 2a_1 x}_{(2a_2 + 2a_0) + (6a_3 + 3a_1)x} + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_{n-2} + 2a_n \right] x^n = 0$

Step 5 $\quad 2a_2 + 2a_0 = 0 \rightarrow a_2 = -a_0$

$\quad 6a_3 + 3a_1 = 0 \rightarrow a_3 = -\dfrac{a_1}{2}$

$\quad (n+2)(n+1) a_{n+2} = -a_{n-2} - (n+2) a_n$

$\quad a_{n+2} = -\dfrac{a_n}{(n+1)} - \dfrac{a_{n-2}}{(n+2)(n+1)}$, $n \geq 2$

$n=2$: $a_4 = -\dfrac{a_2}{3} - \dfrac{a_0}{12} = \dfrac{a_0}{3} - \dfrac{a_0}{12} = \dfrac{a_0}{4}$

$n=3$: $a_5 = -\dfrac{a_3}{4} - \dfrac{a_1}{20} = \dfrac{a_1}{8} - \dfrac{a_1}{20} = \dfrac{3}{40} a_1$

$n=4$: $a_6 = -\dfrac{a_4}{5} - \dfrac{a_2}{30} = -\dfrac{a_0}{20} + \dfrac{a_0}{30} = -\dfrac{1}{60} a_0$

$\vdots \qquad \vdots$

Step 6 $\quad y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots$

$\quad = a_0 + a_1 x - a_0 x^2 - \dfrac{a_1}{2} x^3 + \dfrac{a_0}{4} x^4 + \dfrac{3}{40} a_1 x^5 - \dfrac{1}{60} a_0 x^6 + \ldots$

$\quad = a_0 \left[1 - x^2 + \dfrac{x^4}{4} - \dfrac{x^6}{60} + \ldots \right] + a_1 \left[x - \dfrac{x^3}{2} + \dfrac{3x^5}{40} - \ldots \right] = c_1 y_1 + c_2 y_2$

Note $\quad$ step4 ต้องรวมกลุ่ม ชุด x ก่อน แล้ว $(\blacktriangle)x = 0$

$(\blacktriangle) = 0$

!! อย่าจับ ส.ป.ส. เรียงตัว = 0 !! นะ

(5) $a_1 = -1$

$a_2 = 0$

$a_3 = 1$

$a_{n+3} = 5a_n$

ไม่สมควร

เริ่มคิดจาก $\cancel{a_1}, a_4, a_7$ พิจารณายากส์

a_4, a_7, a_{10} ง่ายกว่า

k = 1
- $n=1$: $a_4 = 5a_1 = -5$
- $n=2$: $a_5 = 5a_2 = 0$
- $n=3$: $a_6 = 5a_3 = 5$

k = 2
- $n=4$: $a_7 = 5a_4 = -5^2$
- $n=5$: $a_8 = 5a_5 = 0$
- $n=6$: $a_9 = 5a_6 = 5^2$

k = 3
- $n=7$: $a_{10} = 5a_7 = -5^3$
- $n=8$: $a_{11} = 5a_8 = 0$
- $n=9$: $a_{12} = 5a_9 = 5^3$

$4, 7, 10, \ldots \rightarrow a_k = 4 + (k-1)3 = 3k+1$

$5, 8, 11, \ldots \rightarrow a_k = 5 + (k-1)3 = 3k+2$

$6, 9, 12, \ldots \rightarrow a_k = 6 + (k-1)3 = 3k+3$

$$\left.\begin{aligned} a_{3k+1} &= -5^k \\ a_{3k+2} &= 0 \\ a_{3k+3} &= +5^k \end{aligned}\right\} k = 1, 2, 3, \ldots$$

$$\therefore\ a_{2011} - a_{2012} = a_{3(670)+1} - a_{3(670)+2}$$

$$= -5^{670} - 0$$