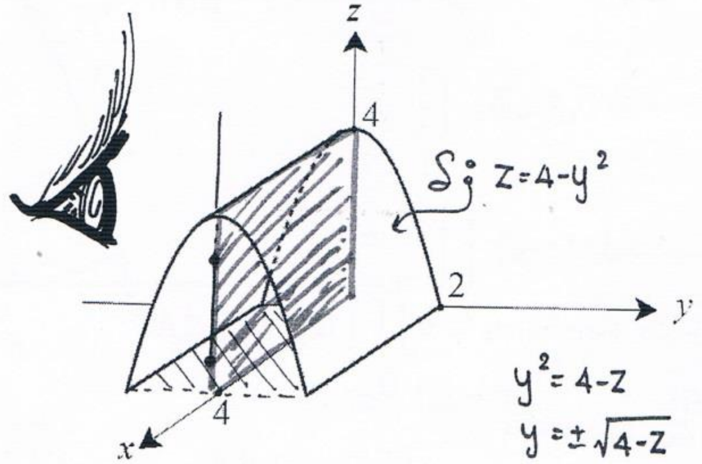
Tutorial : Week XII Surface Integrals of Functions

1. (Midterm II 56) Let S be the surface $z = 4 - y^2$, $0 \leq z \leq 4$ and $0 \leq x \leq 4$. ***Without calculation***, write down the surface integral $\iint\limits_S (x + y^2 + z)\, dS$ as an iterated integral.

ซ้อน

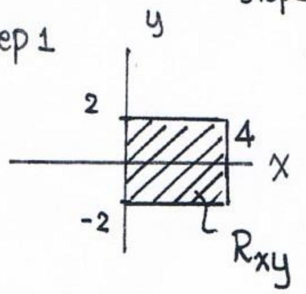


$y^2 = 4 - z$

$y = \pm\sqrt{4-z}$

1.1 When S is projected to the $xy-$plane,

step 1



Step2 $\iint\limits_S (x + y^2 + z)\, dS = \iint\limits_{R_{xy}} (x+y^2+z)\sqrt{z_x^2+z_y^2+1}\, dA$

0   -2y

หลง

1-1

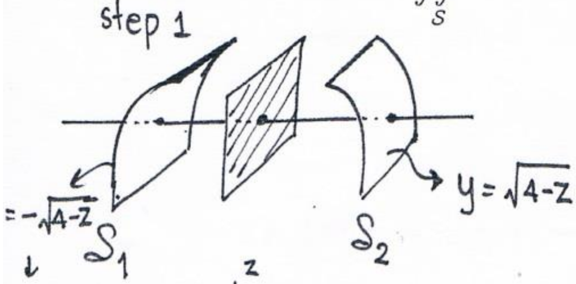
$= \iint\limits_{R_{xy}} (x+\cancel{y^2}+4-\cancel{y^2})\sqrt{4y^2+1}\, dA$

$= \int_{-2}^{2}\int_{0}^{4} (x+4)\sqrt{4y^2+1}\, dx\,dy$ #

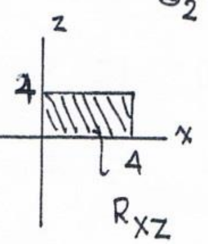
1.2 When S is projected to the $xz-$plane,

$\iint\limits_S (x + y^2 + z)\, dS = \iint\limits_{S_1} (x+y^2+z)\,dS + \iint\limits_{S_2} (x+y^2+z)\,dS$

step 1



$z = \frac{\pm 1}{2\sqrt{4-z}}$



0

$= \iint\limits_{R_{xz}} (x+y^2+z)\sqrt{y_x^2+1^2+y_z^2}\, dA + \iint\limits_{R_{xz}} (x+y^2+z)\sqrt{y_x^2+1+y_z^2}\, dA$

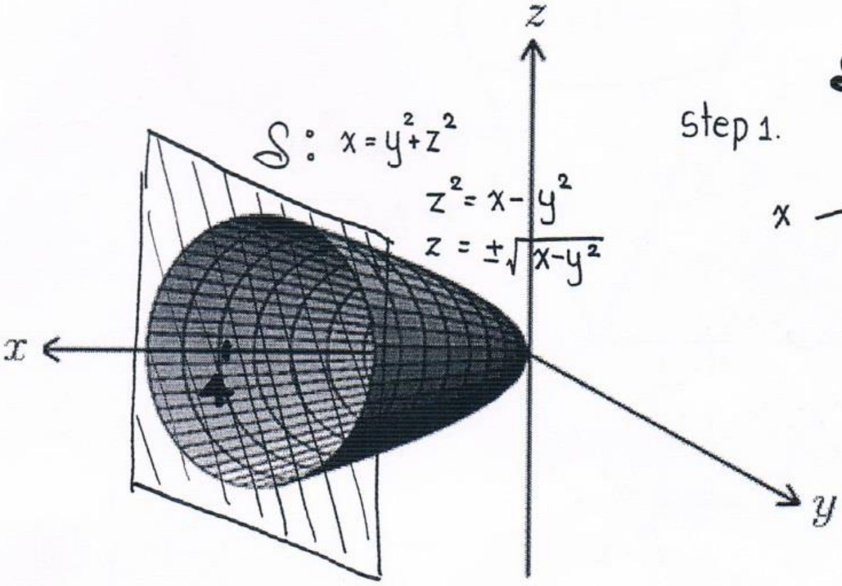
$= \iint\limits_{R_{xz}} (x+4-\cancel{z}+\cancel{z})\sqrt{1+\frac{1}{4(4-z)}}\, dA$ $\times 2$

$= 2\int_{0}^{4}\int_{0}^{4} \frac{(x+4)}{2}\sqrt{\frac{17-4z}{4-z}}\, dx\,dz$ #

2. (**Midterm II 55**) Consider the surface $\mathcal{S}$ which is given by $x = y^2 + z^2$; $0 \le x \le 4$.
Assume that we project the surface $\mathcal{S}$ onto the xy−plane.
Without calculation, set up the surface integral $\iint_{\mathcal{S}} 1+x+y\, dS$ **as an iterated integral.**

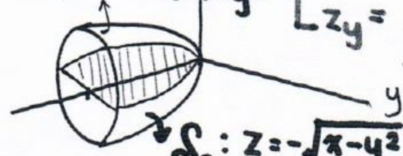
ซ้อน

$\iint \cdots\cdots dxdy$



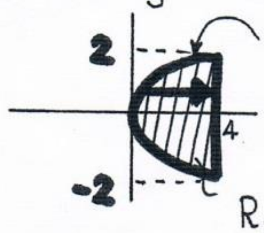
Step 1.

$\mathcal{S}_1 : z = +\sqrt{x-y^2}$ $\quad z_x = \dfrac{1}{2\sqrt{x-y^2}}$, $\quad z_y = \dfrac{-2y}{2\sqrt{x-y^2}}$



$\mathcal{S}_2 : z = -\sqrt{x-y^2}$

(2-1)



$Z=0 : x = y^2$

$4 = y^2$

$y = \pm 2$

Step2 $\displaystyle\iint_{\mathcal{S}} (1+x+y)\,ds = \iint_{\mathcal{S}_1} (1+x+y)\,ds + \iint_{\mathcal{S}_2} (1+x+y)\,ds$ (แยกคิด 2 ผิว เพราะ 2-1 projection)

$$= \iint_{R_{xy}} (1+x+y)\sqrt{z_x^2+z_y^2+1}\; dA + \iint_{R_{xy}} (1+x+y)\sqrt{z_x^2+z_y^2+1}\; dA$$

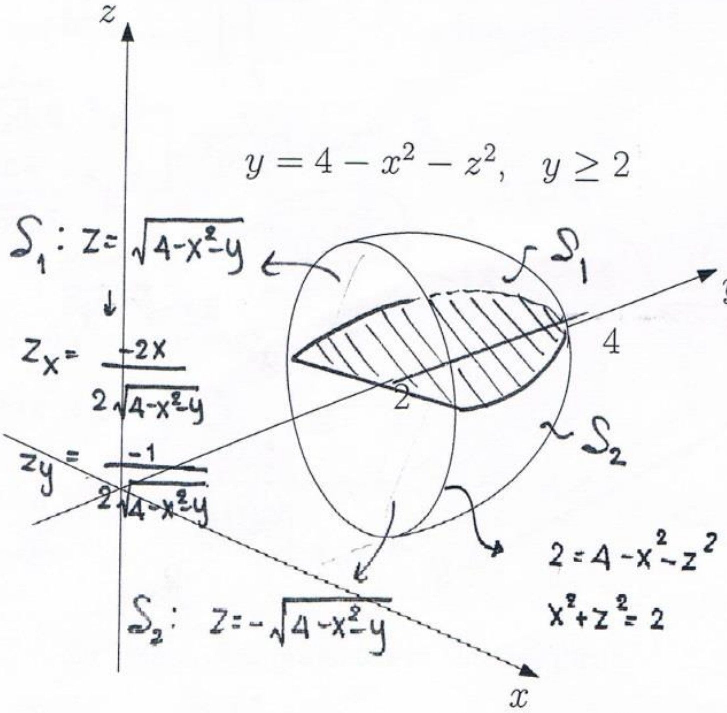
เหมือนด้านหน้า

$$= \mathbf{2}\iint_{R_{xy}} (1+x+y)\sqrt{\frac{1}{4(x-y^2)} + \frac{4y^2}{4(x-y^2)} + 1}\; dA$$

$$\frac{1+4y^2+4(x-y^2)}{4(x-y^2)}$$

$$= 2\int_{-2}^{2}\int_{y^2}^{4} (1+x+y)\sqrt{\frac{1+4x}{4(x-y^2)}}\; dxdy \quad \#$$

3. (**Midterm II** 54) Let $\mathcal{S}$ be the surface $y = 4 - x^2 - y^2$ where $y \geq 2$.
Without calculation, write down the surface integral
$\iint\limits_{\mathcal{S}} (x^2 + 2y + z^2)\, dS$ **as an iterated integral.**



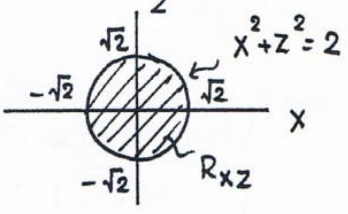
$z^2 = 4 - x^2 - y$

$z = \pm\sqrt{4-x^2-y}$

3.1 When we project $\mathcal{S}$ to the xz−plane.

step1

$$\iint\limits_{\mathcal{S}} (x^2 + 2y + z^2)\, dS = \iint\limits_{R_{xz}} (x^2 + 2\underset{\uparrow \text{หา}}{y} + z^2) \sqrt{\underbrace{y_x^2 + 1 + y_z^2}_{\text{จาก } \mathcal{S}}}\, dA \qquad y_x = -2x,\ y_z = -2z$$



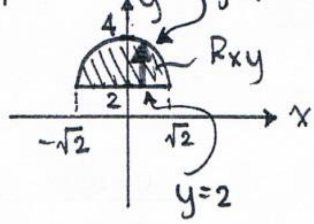
$$= \iint\limits_{R_{xz}} (x^2 + 2(4 - x^2 - z^2) + z^2) \sqrt{4x^2 + 1 + 4z^2}\, dA$$

$$= \iint\limits_{R_{xz}} (8 - x^2 - z^2) \sqrt{4(x^2+z^2)+1}\, dA \quad = \int_0^{2\pi}\int_0^{\sqrt{2}} (8 - r^2)\sqrt{4r^2+1}\, r\,dr\,d\theta \quad \#$$

3.2 When we project $\mathcal{S}$ to the xy−plane.

$$\iint\limits_{\mathcal{S}} (x^2 + 2y + z^2)\, dS = \iint\limits_{\mathcal{S}_1} (x^2 + 2y + z^2)\, dS + \iint\limits_{\mathcal{S}_2} (x^2 + 2y + z^2)\, dS$$

step 1



$$= \iint\limits_{R_{xy}} (x^2 + 2y + z^2)\sqrt{z_x^2 + z_y^2 + 1}\, dA + \iint\limits_{R_{xy}} (x^2 + 2y + z^2)\sqrt{z_x^2 + z_y^2 + 1}\, dA$$

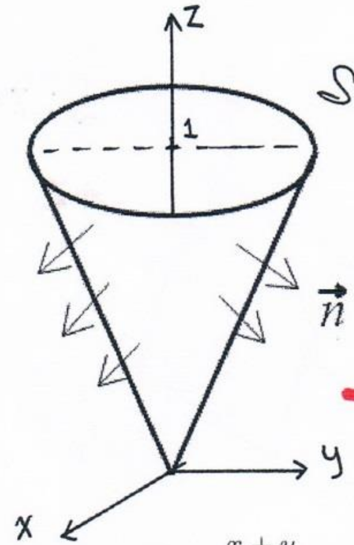
$$= \mathbf{2} \iint\limits_{R_{xy}} (\cancel{x^2} + 2y + 4 - \cancel{x^2} - y) \sqrt{\frac{x^2}{4-x^2-y} + \frac{1}{4(4-x^2-y)} + 1}\, dA$$

$$= 2 \int_{-\sqrt{2}}^{\sqrt{2}} \int_{2}^{4-x^2} (y+4) \sqrt{\frac{17-4y}{4(4-x^2-y)}}\, dy\,dx \quad \#$$

Tutorial : Week XIII Surface Integrals of Vector Fields

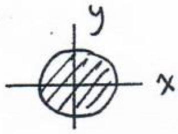
1. (**Midterm II** 56) Let S be the cone $z = \sqrt{x^2+y^2},\ 0 \le z \le 1$.
   Which one is equal to ∇f.

∇g



$S: z = \sqrt{x^2+y^2}$

S1 R_{xy}



S2 Let $g = \sqrt{x^2+y^2} - z$

$$\nabla g = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\vec{i} + \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\vec{j} - \vec{k}$$

$-z$

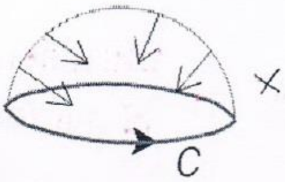
a. $\dfrac{x+y}{\sqrt{x^2+y^2}} + 1$

b. $\dfrac{x+y}{\sqrt{x^2+y^2}} - 1$

c. $\begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \\ -1 \end{bmatrix}$

d. $\begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \\ 1 \end{bmatrix}$

2. (**Midterm II** 56) Which curve is positively oriented with respect to given normal vectors.

a. 

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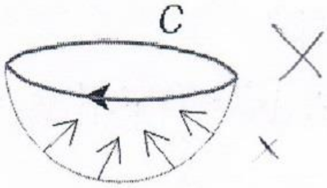
b. 

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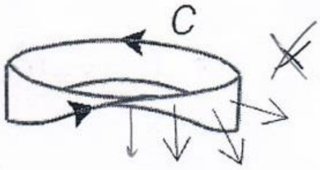
c. 

d. 

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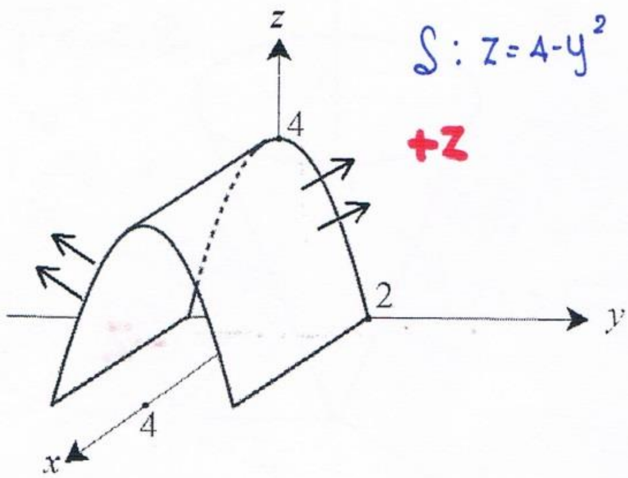
Not Simple Curve

Trick

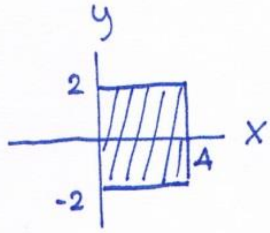
3. (**Midterm II** 56) Let $\vec{F} = xyz\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ and the normal vector be upward-pointing. Compute $\iint_S \vec{F} \cdot d\vec{S}$ where $\mathcal{S}$ is the surface $z = 4 - y^2$, $0 \le z \le 4$ and $0 \le x \le 4$.

step1



$S : z = 4 - y^2$

+z



R_{xy} (1-1)

Step2

$g = z - 4 + y^2$

$\nabla g = 2y\vec{j} + \vec{k}$

$\vec{F} = xyz\vec{i} + y\vec{j} + z\vec{k}$

Step3

ปรับเปลี่ยนตัวแปรใช้ surface

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{R_{xy}} \vec{F} \cdot \nabla g\, dA$$

$$= \iint_{R_{xy}} (2y^2 + z)\, dA = \int_{-2}^{2}\int_{0}^{4} 4 + y^2\, dx\, dy$$

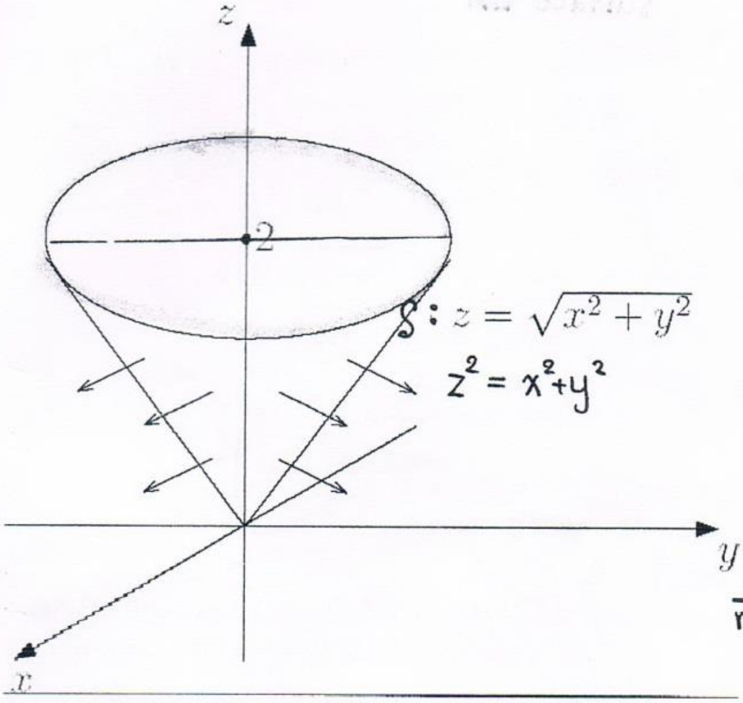
$z = 4 - y^2$

$$= \int_{-2}^{2} (4x + y^2 x)\Big|_0^4 dy = \int_{-2}^{2} 16 + 4y^2\, dy$$

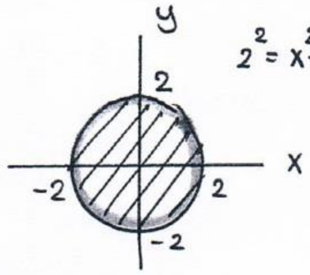
$$= 16y + \frac{4y^3}{3}\Big|_{-2}^{2} = \left(32 + \frac{32}{3}\right) - \left(-32 - \frac{32}{3}\right)$$

$$= \frac{256}{3} \quad \#$$

4. **(Midterm II 54)** Let $\vec{F}(x,y,z) = 3y^2x\vec{i} + 3y^3\vec{j} - 3z^2\vec{k}$ and $\mathcal{S}$ be the cone $z = \sqrt{x^2+y^2}$ where $z \le 2$.
Evaluate $\iint\limits_{S} \vec{F}\cdot d\vec{S}$ under the assumption that the unit normal vector $\vec{n}$ points outward.



step 1 R_{xy}



$2^2 = x^2+y^2$

step 2 Let $g = \sqrt{x^2+y^2} - z$

$\vec{n}$

$$\nabla g = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\vec{i} + \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\vec{j} - \vec{k}$$

$$\vec{F} = 3y^2x\vec{i} + 3y^3\vec{j} - 3z^2\vec{k}$$

step 3

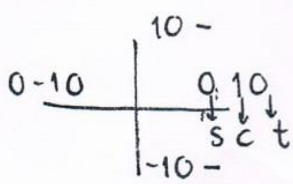
$$\iint\limits_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint\limits_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS = \iint\limits_{R_{xy}} \left(\frac{3y^2x^2}{\sqrt{x^2+y^2}} + \frac{3y^4}{\sqrt{x^2+y^2}} + 3z^2\right) dA$$

$$= \iint\limits_{R_{xy}} \left[3y^2\left(\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right) + 3z^2\right] dA \qquad \{\mathcal{S}: z^2 = x^2+y^2\}$$

$$= \iint\limits_{R_{xy}} \left[3(r\sin\theta)^2 \cdot \frac{r^{\cancel{2}}}{\cancel{r}} + 3r^2\right] dA$$

$$= \int_0^{2\pi}\int_0^2 (3r^3\sin^2\theta + 3r^2)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r^4\sin^2\theta + 3r^3)\,dr\,d\theta$$

J.P.



$$= \int_0^{2\pi} \left(\frac{3r^5}{5}\sin^2\theta + \frac{3r^4}{4}\right)\Big|_0^2 d\theta = \int_0^{2\pi} \frac{96}{5}\sin^2\theta + 12\,d\theta$$

$$= \frac{96}{5}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} + 12\theta\Big|_0^{2\pi} = \frac{96}{5}\left(\frac{\cancel{2}\pi}{\cancel{2}} - \cancel{\frac{\sin 4\pi}{4}}^{\,0}\right) + 24\pi$$

$$= \frac{96}{5}\pi + 24\pi = \frac{216}{5}\pi$$ #