Tutorial XVII : Fourier Series

1. **(Final 56)** The Fourier expansion of a $2\pi-$periodic function is based on the <u>orthonormal set</u>

$$\mathcal{N} = \left\{ \frac{1}{\sqrt{2}},\ \cos nx,\ \sin nx\ \ : n = 1, 2, 3, \ldots \right\}$$

1.1 $\displaystyle\int_{-\pi}^{\pi} (\cos 20x)(\sin 30x)\, dx =$ ___0___

1.2 $\displaystyle\int_{-\pi}^{\pi} (\cos 20x)(\cos 30x)\, dx =$ ___0___

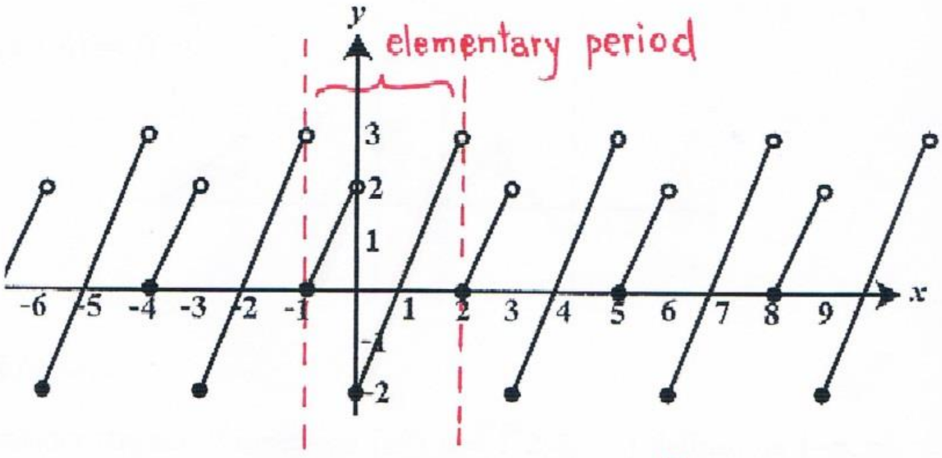
1.3 $\displaystyle\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}}(\cos 20x)(\cos 20x)\, dx =$ ___$\frac{\pi}{\sqrt{2}}$___

1.4 $\displaystyle\int_{-\pi}^{\pi} (\sin 30x)(\sin 30x)\, dx =$ ___π___

2. **(Final 56)**

2.1 The elementary period of the function $f(x) = \cos\left(\frac{x}{4}\right)$ is ___$\frac{2\pi}{\left(\frac{1}{4}\right)} = 8\pi$___.

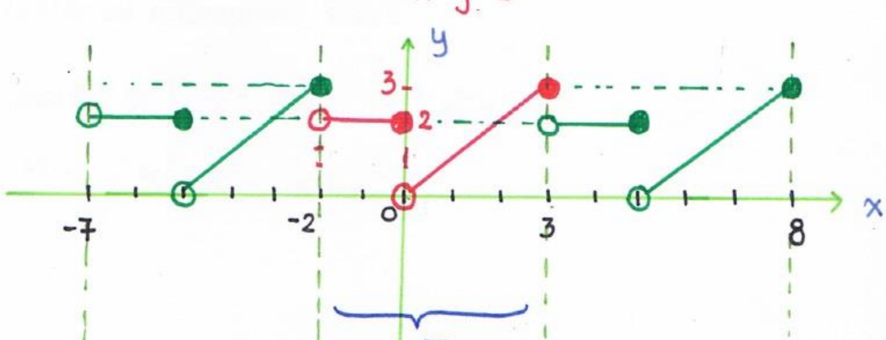
2.2 Let the graph of a function $f(x)$ be given as follows.



The elementary period of the function $f(x)$ is ___3___.

2.3 Sketch the graph of $f(x) = \begin{cases} x, & 0 < x \leq 3; \\ 2, & -2 < x \leq 0 \end{cases}$ T=5

and $f(x+5) = f(x)$. ↳ คงที่ y=2



3. **(Final 55)**

3.1 Consider the set of functions $\mathcal{A} = \{x^n : n = 1, 2, 3, \dots\}$ defined on $[-\pi, \pi]$.

3.1.1 Choose two functions x^m, $x^n \in \mathcal{A}$ such that x^m is orthogonal to x^n. ($x^m \perp x^n$)

$\langle x^m, x^n \rangle = 0 \rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} x^m \cdot x^n \, dx = 0 \quad \therefore \; x^m = x^2 \; , \; x^n = x^3$

↳ odd

3.1.2 Choose two functions x^m, $x^n \in \mathcal{A}$ such that x^m is not orthogonal to x^n. ($x^m \not\perp x^n$)

$\int_{-\pi}^{\pi} x^m \cdot x^n \, dx \neq 0 \quad \therefore \; x^m = x^2 \; , \; x^n = x^4$

↳ even

3.1.3 Is $\mathcal{A}$ an orthogonal set? Why?

No. เพราะมี x^m, x^n ซึ่ง $\langle x^m, x^n \rangle \neq 0$

3.2 Let $\mathcal{B} = \{f_n(x) : n = 1, 2, 3, \dots\}$ be an orthogonal set.

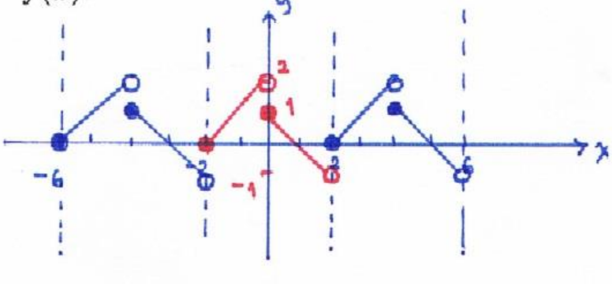
Then $\langle f_{2555}, f_{2012} \rangle =$ 0 .

4. **(Final 55)**Sketch the graph of

$$f(x) = \begin{cases} x + 2, & -2 \le x < 0; \\ 1 - x, & 0 \le x < 2 \end{cases}$$

x	y
-2	0
0	2
0	1
2	-1

and $f(x+4) = f(x)$.



5. **(Final 54)**

5.1 Consider the set of functions $\{x^n : n = 1, 2, 3, \dots\}$ defined on $[-\pi, \pi]$.

5.1.1 Is x^n orthogonal to x^{n+1}? Why? Yes.

$\langle x^n, x^{n+1} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n+1} \, dx = 0$

↳ odd

5.1.2 Is this set orthogonal? Why?

No. เพราะมี x^m, x^n ซึ่ง $\langle x^m, x^n \rangle \neq 0$

เช่น $x^m = x^2$, $x^n = x^6$

ค่าคงที่ $\in I^+$

5.2 Let n be a positive integer, $f(x) = x^n$, $g(x) = x^{2n}$, $h(x) = \sin 4x$ and $k(x) = \cos nx$ be functions defined on $[-\pi, \pi]$.

5.2.1 Compute $\langle g, h\rangle$. $= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^{2n}}_{\text{even}} \cdot \underbrace{\sin 4x}_{\text{odd}}\, dx = 0$

5.2.2 Find a condition on n such that $\langle f, k\rangle = 0$.

$\frac{1}{\pi}\int_{-\pi}^{\pi} x^n \cdot \underbrace{\cos nx}_{\text{even}}\, dx = 0$

thus ; x^n ต้องเป็น odd f^n

$\therefore\ n \in I^+_{odd}$

6. (**Final** 54) The elementary period of $f(x) = \sin 8x$ is $\frac{2\pi}{8} = \frac{\pi}{4}$.

1) $a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$

$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\, dx$

$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\, dx$

Tutorial XVIII : Fourier Series

1. (Final 56) Let $f(x)$ be a periodic function with period 4. The Fourier series of f is given by

$$f(x) \sim \frac{3}{4} + \frac{4}{\pi^2}\left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}\cos\frac{n\pi x}{2} + \frac{1}{(2n-1)^2}\sin\frac{n\pi x}{2}\right]$$

$T = 4$

$L = 2$

$\frac{a_0}{2} = \frac{3}{4} \rightarrow a_0 = \frac{3}{2}$

$a_n = \frac{4}{\pi^2}\frac{(-1)^n}{n^2+1}$

$b_n = \frac{4}{\pi^2}\frac{1}{(2n-1)^2}$

$f(x+4) = f(x)$

$f(x+8) = f(x+4) = f(x)$

1.1 $\int_{-2}^{2} f(x)\,dx =$ $La_0 = 2a_0 = 2\left(\frac{3}{2}\right) = 3$

1.2 $a_3 =$ $\frac{4}{\pi^2}\frac{(-1)^3}{3^2+1} = \frac{-2}{5\pi^2}$

1.3 $b_7 =$ $\frac{4}{\pi^2}\frac{1}{(2\cdot7-1)^2} = \frac{4}{169\pi^2}$

1.4 $\int_{-2}^{2} f(x)\cos\pi x\,dx =$ $La_n = La_2 = 2\left(\frac{4}{\pi^2}\right)\frac{(-1)^2}{4+1} = \frac{8}{5\pi^2}$ $\hookrightarrow n=2$

1.5 $\int_{-2}^{2} f(x)\sin\frac{3\pi x}{2}\,dx =$ $Lb_n = Lb_3 = 2\left(\frac{4}{\pi^2}\right)\frac{1}{5^2} = \frac{8}{25\pi^2}$ $\hookrightarrow n=3$

1.6 $\int_{-2}^{2} f(x+8)\,dx =$ $\int_{-2}^{2} f(x)\,dx = 3$

2. (Final 56) The Fourier series of $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 2x, & 0 < x < 1 \end{cases}$ and $f(x+2) = f(x)$

is given by

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x.$$

2.1 (4 points) Evaluate $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots$.

Consider $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ — **.

ดังนั้น $\cos(2n-1)\pi x = 1$, $\sin n\pi x = 0$

$\boxed{x=0}$; $\frac{f(0^+)+f(0^-)}{2} = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cancel{\cos 0}^{\,1}$

ไม่ต่อเนื่อง

$0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{2}\times\frac{\pi^2}{4} = \frac{\pi^2}{8}$ #

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2.2 (4 points) Evaluate $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$.

$\boxed{x = \frac{1}{2}}$; $2\left(\frac{1}{2}\right) = \frac{1}{2} - 0 + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2}$

ต่อเนื่อง

$$1 = \frac{1}{2} + \frac{2}{\pi}\Big[\underset{n=1}{\frac{1}{1}} + \underset{n=2}{0} + \underset{n=3}{\frac{1}{3}(-1)} + \underset{n=4}{0} + \underset{n=5}{\frac{1}{5}} + \overset{n=6}{0} + \underset{n=7}{\frac{1}{7}(-1)} + \dots\Big]$$

$$\therefore \quad \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{2}\left(1 - \frac{1}{2}\right) = \frac{\pi}{4} \quad \#$$

~~3.~~ (Final 56) <u>True of False</u>.
(correct 2 points, no answer -0.5 point, wrong -1 point)

3.1 __F__ If f is an odd function and g is an even function, then $\int_{-5}^{5} f(x) + g(x)\,dx = 0.$ → $\int_{-5}^{5} \cancel{f(x)}^{\,0} dx + \int_{-5}^{5} g(x)\,dx \neq 0$ ↳even

3.2 __F__ If f is an odd function and g is an even function, then $\int_{-5}^{5} f(x)g(x)\,dx = 2\int_{0}^{5} f(x)g(x)\,dx.$
↳odd ↳even → odd

3.3 __T__ If f is a periodic odd function, then the Fourier series of f is a Fourier sine.

↗ หาได้เสมอ

3.4 __T__ If $x = x_0$ is an ordinary point of the equation $p(x)y'' + q(x)y' + s(x)y = 0$, then both $\lim_{x \to x_0}(x - x_0)\frac{q(x)}{p(x)}$ and $\lim_{x\to x_0}(x - x_0)^2\frac{s(x)}{p(x)}$ exist.

3.5 __T__ If $x = x_0$ is an irregular singular point and $\lim_{x\to x_0}(x - x_0)\frac{q(x)}{p(x)}$ exists, then $\lim_{x\to x_0}(x - x_0)^2\frac{s(x)}{p(x)}$ does not exist.

3.6 __T__ $\sum_{n=1}^{\infty} a_n(n-1)x^{n+3} = \sum_{n=-1}^{\infty} a_{n+2}(n+1)x^{n+5}.$

3.7 __F__ If the Fourier series of f is $a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$, then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$ ✗ $a_0 = \frac{(a_0)_F}{2}$

3.8 __T__ If f is periodic and has period p, then $3p$ is a period of f.

$np,\ n \in I^+$

4. **(Final** 56) Match the following function with their Fourier series

(a) $2\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}n^2\pi^2+6(-1)^n}{n^3}\sin nx$ F. sin → odd

(b) $1-\dfrac{8}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\left(\cos\dfrac{(2n-1)\pi x}{2}\right)$ F. cos → even

↳ L=2

(c) $\dfrac{1}{2}+\sum_{n=1}^{\infty}\dfrac{2}{(2n-1)\pi}\sin(2n-1)x$ F. sin → odd

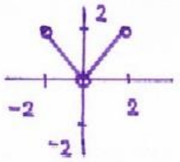
(d) $\dfrac{7}{2}+2\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n-1}{(\pi n)^2}\cos n\pi x+\dfrac{2(-1)^n-3}{n\pi}\sin n\pi x\right]$ not odd , even

↳ L=1

Write down (a),(b), (c) or (d)

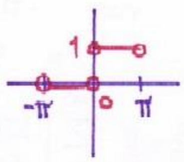
4.1 $f(x)=\begin{cases}-x, & -2<x<0\\ x, & 0<x<2\end{cases}$
and $f(x+4)=f(x)$

Answer b ↳ L=2



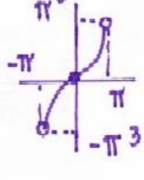
4.2 $f(x)=\begin{cases}0, & -\pi<x<0\\ 1, & 0<x<\pi\end{cases}$
and $f(x+2\pi)=f(x)$

Answer c



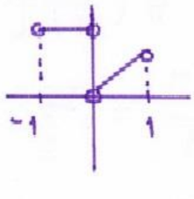
4.3 $f(x)=x^3;\ -\pi<x<\pi$ and $f(x+2\pi)=f(x)$

Answer a



4.4 $f(x)=\begin{cases}6, & -1<x<0\\ 2x, & 0<x<1\end{cases}$
and $f(x+2)=f(x)$.

Answer d → T=2 L=1



n=1 n=2 n=3

(6.3) $x=\frac{1}{2}$: $f\left(\frac{1}{2}\right) = \frac{7}{2} + 2\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{(\pi n)^2}\cos\frac{n\pi}{2} + \frac{2(-1)^n-3}{n\pi}\sin\frac{n\pi}{2}\right]$

$$1 = \frac{7}{2} + 2\sum_{n=1}^{\infty}\frac{2(-1)^n-3}{n\pi}\sin n\frac{\pi}{2}$$

$$1 = \frac{7}{2} + \frac{2}{\pi}\left[\frac{-5}{\pi}(1) + 0 + \frac{5}{3\pi} + 0 - \frac{5}{5\pi} + \dots\right]$$

n=1 n=2 n=3

$$1-\frac{7}{2} = \frac{2}{\pi}\left(\frac{-5}{\pi}\right)\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\dots\right]$$

$$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{-5}{2}\times\frac{\pi^2}{-10} = \frac{\pi^2}{4}$$

5. (**Final** 55)Let f be a periodic function whose **period is equal to** π. Then the formula for the Fourier series f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right)$$

$$\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2nx + b_n \sin 2nx\right) \quad \#$$

$T = \pi \rightsquigarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$L = \frac{\pi}{2}$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx\,dx$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin 2nx\,dx \quad \#$$

6. (**Final** 55)The Fourier series of

$$f(x) = \begin{cases} 6, & -1 < x < 0; \\ 2x, & 0 < x < 1 \end{cases}$$

and $f(x+2) = f(x)$ is given by

$$f(x) \sim \frac{7}{2} + 2\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(\pi n)^2}\cos n\pi x + \frac{2(-1)^n - 3}{n\pi}\sin n\pi x\right].$$

Use this series to compute the following quantities.

6.1 $\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{(\pi n)^2} =$ $\frac{1}{4}$

6.2 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} =$ $\frac{\pi^2}{8}$

$\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

6.3 $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} =$ $\frac{\pi}{4}$

$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

6.1 $x = 0$: $\displaystyle\frac{f(0^+) + f(0^-)}{2} = \frac{7}{2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{(\pi n)^2}\cos 0$

$\displaystyle\frac{0+6}{2} = \frac{7}{2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{(\pi n)^2}$

$\displaystyle\therefore \sum_{n=1}^{\infty}\frac{1-(-1)^n}{(\pi n)^2} = \frac{1}{2}\left[\frac{7}{2} - 3\right] = \frac{1}{4}$

6.2 $\displaystyle\frac{2}{\pi^2 \cdot 1} + 0 + \frac{2}{\pi^2 3^2} + 0 + \frac{2}{\pi^2 5^2} + \ldots = \frac{1}{4}$

$\displaystyle\left(\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) = \frac{1}{4} \times \frac{\pi^2}{2}$

7. (**Final** 55) Match the following functions with their Fourier series.

$f(x)$

(a) $\displaystyle\pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$ F. sine → odd

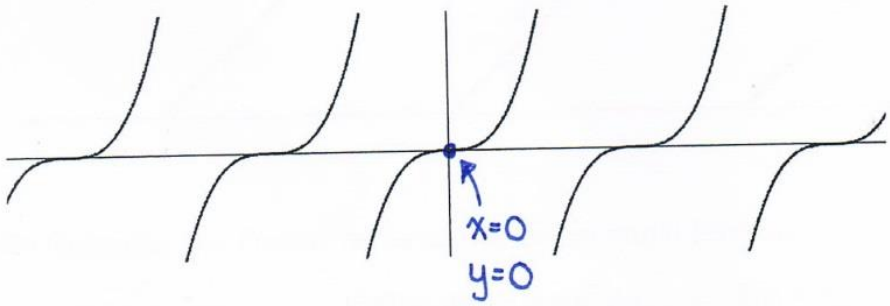
(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)(-1)^n \sin nx$ F. sine → odd $x = 0 \rightarrow f(x) = 0$

(c) $\displaystyle\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$ F. cos → even

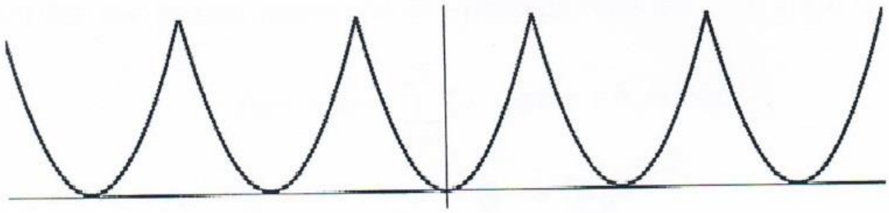
(d) $\displaystyle\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$ F. → not odd, even

Write down (a), (b), (c), or (d).

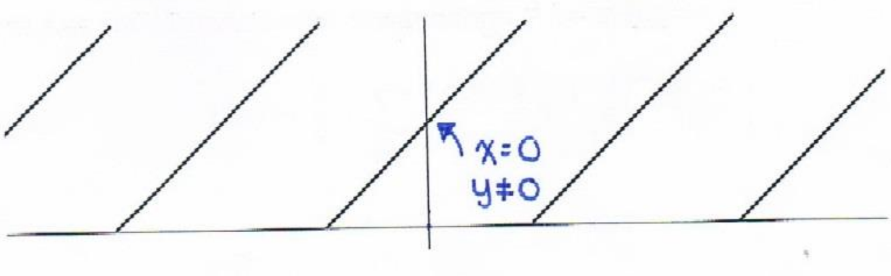
7.1 Answer b



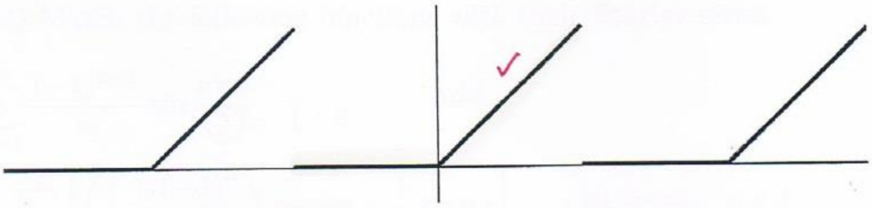
7.2 Answer c



7.3 Answer a



7.4 Answer d



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8. (**Final** 55) Compute the Fourier series of the $2\pi-$periodic function

$$f(x) = \sin 4x + \sin^2 4x. \quad = \sin 4x + \frac{1-\cos 2(4x)}{2}$$

Answer $f(x) \sim$ $\frac{1}{2} + \sin 4x - \frac{1}{2}\cos 8x$

9. (**Final** 54) Let the Fourier series of a $2\pi-$periodic function f be given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

$a_0 = \frac{A_0}{2}$ ปกติ

Then $a_0 =$ $\frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\right]$

and $a_n =$ $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ $b_n =$ $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ when $n = 1, 2, 3, \ldots$.

10. (**Final** 54) Let the Fourier series of a function f be given by

$$f(x) \sim \frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n+1} + \frac{(-1)^n \sin nx}{(2n-1)^2}\right).$$

$\frac{a_0}{2} = \frac{3}{4} \rightarrow a_0 = \frac{3}{2}$

$a_n = \frac{1}{n+1}$

$b_n = \frac{(-1)^n}{(2n-1)^2}$

Compute

10.1 $\int_{-\pi}^{\pi} f(x)\,dx =$ $\frac{3\pi}{2}$ #

10.2 $\int_{-\pi}^{\pi} f(x)\sin \underset{n=5}{5}x\,dx =$ $\pi b_n = \pi b_5 = \frac{\pi(-1)^5}{(2\cdot5-1)^2} = -\frac{\pi}{81}$ #

10.3 $\int_{-\pi}^{\pi} f(x)\cos \underset{n=7}{7}x\,dx =$ $\pi a_n = \pi a_7 = \pi\left(\frac{1}{7+1}\right) = \frac{\pi}{8}$ #

10.4 $\int_{-\pi}^{\pi} f(x)\sin^2 2x\,dx =$ $\int_{-\pi}^{\pi} f(x)\left[\frac{1-\cos 4x}{2}\right]dx = \frac{1}{2}\left[\int_{-\pi}^{\pi} f(x)\,dx - \int_{-\pi}^{\pi} f(x)\cos 4x\,dx\right]$

$\pi a_n = \pi a_4$

$= \frac{1}{2}\left[\frac{3}{2}\pi - \pi\cdot\frac{1}{(4+1)}\right] = \frac{13\pi}{20}$ #

10.5 $\int_{-\pi}^{\pi} f(x+6\pi)\,dx =$ $\int_{-\pi}^{\pi} f(x)\,dx = \frac{3}{2}\pi$ #

ทุกๆ 6π graph เหมือนเดิม ∵ ทุกๆ 2π graph เดิม

∴ ทุกๆ 6π graph เดิม

~~11~~. (**Final** 54) Match the following functions with their Fourier series.

✓ (a) $\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$ $\rightarrow L=2$ odd

(b) $\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\left(\frac{1+(-1)^n}{\pi n^2}\right)\cos nx+\frac{1}{n}\sin nx\right]$ not even, odd

(c) $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$ even

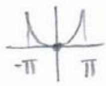
(d) $\frac{\pi+2}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$ even

Write down (a), (b), (c), or (d).

11.1 $f(x)=1+|x|$ when $x\in(-\pi,\pi)$ and $f(x+2\pi)=f(x)$. Answer d

$x=0, y=1$

11.2 $f(x)=x^2$ when $x\in(-\pi,\pi)$ and $f(x+2\pi)=f(x)$. Answer C



$-\pi \quad \pi$

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{x^3}{3}dx=\frac{2}{\pi}\cdot\frac{\pi^3}{3}=\frac{2\pi^2}{3}$

11.3 $f(x)=x$ when $x\in(-2,2)$ and $f(x+4)=f(x)$. Answer a $\rightarrow L=2$

11.4 $f(x)=\begin{cases}0, & -\pi\le x<0;\\ \pi-x, & 0\le x<\pi\end{cases}$ and $f(x+2\pi)=f(x)$. Answer b

12. (**Final** 54) The Fourier series of $f(x)=\frac{\pi}{2\sinh\pi}e^{3x}$ when $x\in(-\frac{\pi}{3},\frac{\pi}{3})$ and $f(x+\frac{2\pi}{3})=f(x)$ is given by

$$f(x)\sim\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos 3nx-n\sin 3nx).$$

Use this series to compute the following quantities.
Note : $\frac{\pi}{2\sinh\pi}$ is a constant.

$\boxed{x=0}$: $f(0)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos 0$

$\frac{\pi}{2\sinh\pi}e^0=\frac{1}{2}+\left[-\frac{1}{1+1}+\frac{1}{1+2^2}-\frac{1}{1+3^2}+\dots\right]$

12.1 $\frac{1}{2}-\frac{1}{1+1}+\frac{1}{1+4}-\frac{1}{1+9}+\frac{1}{1+16}-\cdots=$ $\frac{\pi}{2\sinh\pi}$ #

12.2 $\frac{1}{2}+\frac{1}{1+1}+\frac{1}{1+4}+\frac{1}{1+9}+\frac{1}{1+16}+\cdots=$ $\frac{\pi e^{\pi}}{2\sinh\pi}$ #

$\boxed{x=\frac{\pi}{3}}$: $f(\frac{\pi}{3})=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos(3n\frac{\pi}{3})$

$\frac{\pi}{2\sinh\pi}e^{\frac{3\pi}{3}}=\frac{1}{2}+\frac{1}{1+1}+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\dots$

13. (**Final** 54) Compute the Fourier series of the 2π−periodic function

$$f(x) = \frac{5}{2} - (\cos 3x)(\cos 7x) - (\sin 3x)(\sin 7x).$$

Answer $f(x) \sim$ $\frac{5}{2} - \cos 4x$ #

พิจารณา $(\cos 3x)(\cos 7x)$

$$\frac{2}{2}\cos(7x)\cos(3x) = \frac{1}{2}[\cos(7x+3x)+\cos(7x-3x)]$$

$$= \frac{1}{2}[\cos 10x + \cos 4x] \quad -*$$

พิจารณา $(\sin 3x)(\sin 7x)$

$$\frac{2}{2}(\sin 7x)\sin(3x) = \frac{1}{2}[\cos(7x-3x)-\cos(7x+3x)]$$

$$= \frac{1}{2}[\cos 4x - \cos 10x] \quad -**$$

$$f(x) = \frac{5}{2} - \cancel{\frac{1}{2}\cos 10x} - \frac{1}{2}\cos 4x - \frac{1}{2}\cos 4x + \cancel{\frac{1}{2}\cos 10x}$$

$$= \underbrace{\frac{5}{2}}_{\frac{a_0}{2}} - \underbrace{\cos 4x}_{a_4}$$

เป็น f^n ในรูป Trigone แล้ว
จึงเรียกว่าเป็น Fourier series ได้เลย
มี 2 พจน์

Note ต้องเป็นพจน์ Trigone กำลัง 1

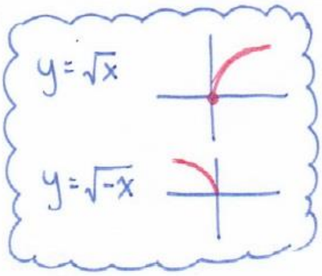
Tutorial XIX : Fourier Series

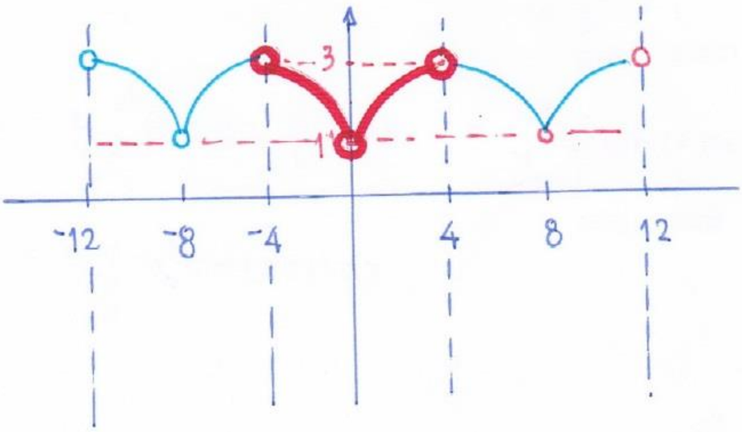
1. **(Final** 56) Let $f(x) = 1 + \sqrt{x}$ when $0 < x < 4$.

1.1 Find the **even periodic** extension of f and plot the graph of the extended function.

$\hookrightarrow f(-x)$

$$F(x) = \begin{cases} 1+\sqrt{x} & ; \ 0<x<4 \\ 1+\sqrt{-x} & ; \ -4<x<0 \end{cases}, \quad F(x+8) = F(x)$$

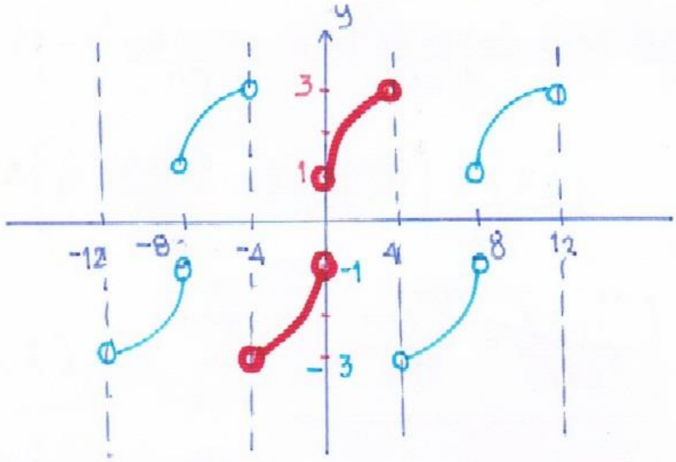




1.2 Find the **odd periodic** extension of f and plot the graph of the extended function.

$\hookrightarrow -f(-x)$

$$F(x) = \begin{cases} 1+\sqrt{x} & ; \ 0<x<4 \\ -(1+\sqrt{-x}) & ; \ -4<x<0 \end{cases}, \quad F(x+8) = F(x)$$



→ f^n คี่ จะได้ F.S.

2. (**Final** 56) Find the Fourier series of the function $f(x) = x^3$ where $-1 < x < 1$ and $f(x+2) = f(x)$.

$L = 1$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{\cancel{L}_1} x + b_n \sin \frac{n\pi}{\cancel{L}_1} x \right)$$

1) หา $a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx = \frac{1}{1} \int_{-1}^{1} \underbrace{x^3}_{\text{odd}}\,dx = 0$

2) หา $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{\cancel{L}_1} x\,dx = \int_{-1}^{1} \underbrace{x^3}_{\text{odd}} \underbrace{\cos(n\pi x)}_{\text{even}}\,dx = 0$

3) หา $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\,dx = \int_{-1}^{1} \underbrace{x^3}_{\text{odd}} \underbrace{\sin(n\pi x)}_{\text{odd}}\,dx$ → even

$= 2 \int_0^1 x^3 \sin(n\pi x)\,dx$

u	v	
x^3	$\sin(n\pi x)$	
$3x^2$	$\frac{-\cos(n\pi x)}{n\pi}$	+
$6x$	$\frac{-\sin(n\pi x)}{(n\pi)^2}$	-
6	$+\frac{\cos(n\pi x)}{(n\pi)^3}$	+
0	$\frac{\sin(n\pi x)}{(n\pi)^4}$	-

$$= 2\left[-x^3 \frac{\cos(n\pi x)}{n\pi} + 3x^2 \frac{\sin(n\pi x)}{(n\pi)^2} + 6x \frac{\cos(n\pi x)}{(n\pi)^3} - \frac{6 \sin(n\pi x)}{(n\pi)^4} \right]_0^1$$

$$= 2\left[\left(-\frac{\cos(n\pi)}{n\pi} + \frac{6\cos(n\pi)}{(n\pi)^3} \right) - (0) \right] = 2\left[\frac{-(-1)^n}{n\pi} + \frac{6(-1)^n}{(n\pi)^3} \right]$$

$$\therefore \underbrace{f(x)}_{x^3} \sim \sum_{n=1}^{\infty} \underbrace{2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{6(-1)^n}{(n\pi)^3} \right]}_{b_n} \sin(n\pi x)$$

#

3. (**Final** 56) Let $f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1 - x^2, & -1 < x < 0. \end{cases}$

Extend the function and find the Fourier cosine series.

↳ even เพิ่ม $f(-x)$

$$F(x) = \begin{cases} 0 & , -2 < x < -1 \\ 1-x^2 & , -1 < x < 0 \\ 1-(-x)^2 & , 0 < x < 1 \\ 0 & , 1 < x < 2 \end{cases}$$

เป็น even f^{n}

$F(x+4) = F(x)$

$T = 4$, $L = 2$

Ans 1

even f^{n} : $b_n = 0$

$$f_e(x) = F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi}{L}x + \cancel{b_n \sin\frac{n\pi}{L}x} \right]$$

(1) $a_0 = \frac{1}{L}\int_{-L}^{L} F(x)\,dx = \frac{2}{L}\int_{0}^{L} F(x)\,dx$

$$= \frac{2}{2}\left[\int_0^1 (1-x^2)\,dx + \cancel{\int_1^2 0\,dx} \right] = \left(x - \frac{x^3}{3}\right)\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

(2) $a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\frac{n\pi}{L}x\,dx = \frac{2}{2}\left[\int_0^1 (1-x^2)\cos\frac{n\pi}{2}x\,dx + \cancel{\int_1^2 0\,dx} \right]$

$$= \int_0^1 \cos\frac{n\pi}{2}x\,dx - \int_0^1 x^2 \cos\frac{n\pi}{2}x\,dx$$

u	v
x^2	$\cos\frac{n\pi}{2}x$
$2x$	$\frac{2}{n\pi}\sin\frac{n\pi}{2}x$ ↘ +
2	$-\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi}{2}x$ ↘ −
0	$-\left(\frac{2}{n\pi}\right)^3\sin\frac{n\pi}{2}x$ ↘ +

$$= \frac{\sin\frac{n\pi}{2}x}{\left(\frac{n\pi}{2}\right)}\Bigg|_0^1 - \left[\frac{2x^2}{n\pi}\sin\frac{n\pi}{2}x + 2x\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi}{2}x - 2\left(\frac{2}{n\pi}\right)^3\sin\frac{n\pi}{2}x \right]_0^1$$

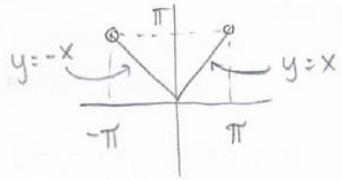
$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} - \left[\left(\frac{2}{n\pi}\sin\frac{n\pi}{2} + 2\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi}{2} - 2\left(\frac{2}{n\pi}\right)^3\sin\frac{n\pi}{2} \right) - 0 \right]$$

$$= -2\left(\frac{2}{n\pi}\right)^2 \cos\frac{n\pi}{2} + 2\left(\frac{2}{n\pi}\right)^3 \sin\frac{n\pi}{2}$$

$$\therefore \quad f_e(x) = F(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} 2\left(\frac{2}{n\pi}\right)^2 \left[\frac{2}{n\pi}\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2} \right] \cos\frac{n\pi}{2}x$$

#

4. (**Final** 55)Find the Fourier series of the function $f(x) = |x|$ where $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$.



$$f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$f(x)$ เป็น even $f^{\underline{n}}$ $T = 2\pi$, $L = \pi$

even $f^{\underline{n}}$: $b_n = 0$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + \cancel{b_n \sin nx}]$$

(1) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{x^2}{2}\Big|_0^{\pi} = \pi$

(2) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$

u	v
x	$\cos nx$
1	$\frac{\sin nx}{n}$ → +
0	$-\frac{\cos nx}{n^2}$ → −

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{n}\cancel{\sin n\pi}^{\,0} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\begin{cases} 0, & \text{even} \\ \frac{-4}{n^2\pi}, & \text{odd} \end{cases}$$

$$\therefore \quad f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^2}\cos nx$$

#

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{1}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos[(2k-1)x]$$